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A THEORETICAL INVESTIGATION OF THE LATERAL OSCILLATIONS OF AN AIRPLANE WITH FREE RUDDER WITH SPECIAL REFERENCE TO THE EFFECT OF FRICTION

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SUMMARY

Charts showing the variation in dynamic stability with the rudder hinge-moment characteristics are presented. A stabilizing rudder floating tendency combined with a high degree of aerodynamic balance is shown to lead to oscillations of increasing amplitude. This dynamic instability is increased by viscous friction in the rudder control system.

The presence of solid friction in the rudder control system will cause steady oscillations of constant amplitude if the floating angle of the rudder per unit angle of sideslip is stabilizing and greater than a certain critical value that depends on other airplane parameters, such as vertical-tail area and airplane moment of inertia about the vertical axis. The amplitude of the steady oscillation is proportional to the amount of friction and is generally quite small but increases as the condition of dynamic instability is approached.

An approximate method of calculating the amplitudes of the steady oscillation is explained and is illustrated by a numerical example. A more exact step-by-step calculation of the motion is also made and it is shown that the agreement with the approximate method is good.

INTRODUCTION

Flight tests have shown that, under certain conditions of rudder balance, undamped lateral oscillations may occur when the control is freed. The oscillations involve coupling between yawing motions of the airplane and movements of the rudder and depend on the amount of friction in

the rudder control system. A previous theoretical investigation (reference 1) showed the existence of these unstable oscillations but did not cover a sufficiently large range of the variables, particularly of the rudder floating-moment parameter. The importance of this parameter has been emphasized by the recent interest in control surfaces having a positive, or stabilizing, floating tendency - that is, surfaces the free movements of which tend to oppose any disturbance of the airplane.

Theoretical calculations (reference 2) have shown that, for a rudder having a stabilizing floating tendency, increases in rudder damping may cause unstable oscillations. A general discussion of the effect of friction in producing oscillations of limited magnitude under these conditions is given in an unpublished document by Schairer and Bush of Boeing Aircraft Co.

Because of the advantages in using a rudder with a positive floating tendency, the undamped oscillations that may occur when such a surface is used have been fully investigated. The present report gives the results of a theoretical investigation of the subject and deals primarily with the effects of friction on the stability of the oscillations in yaw of an airplane with rudder free. The effects of rudder inertia and mass balance, airplane inertia, weathercock stability, and rudder effectiveness are also treated.

SYMBOLS

b wing span

C_h hinge-moment coefficient $\left(H/\frac{\rho}{2}V^2 S_r c_r\right)$

C_{h_f} frictional hinge-moment coefficient $\left(H_f/\frac{\rho}{2}V^2 S_r c_r\right)$

$C_{n_\delta} = \frac{\partial C_n}{\partial \delta}$; $C_{h_{D\delta}} = \frac{\partial C_h}{\partial D\delta}$; and so forth

$C_{h_{D\delta_f}}$ effective increment in viscous-damping coefficient due to solid friction

c_r rudder chord

D differential operator (d/ds)

H hinge moment

H_f	frictional hinge moment
k_r	radius of gyration of rudder about hinge axis, divided by semispan
k_Z	radius of gyration of airplane about vertical axis, divided by semispan
l	tail length divided by wing semispan
m	mass of airplane
m_r	mass of rudder
s	distance traveled in semispans ($2Vt/b$)
S	wing area
S_r	rudder area
t	time
V	airspeed
x_r	distance of rudder center of gravity behind hinge, divided by semispan
α_e	effective angle of attack of vertical tail
δ	angle of rudder deflection measured from neutral position, radians
$\bar{\delta}$	amplitude of rudder oscillation
ϵ	angle of lag (angle between the position of the rudder and the position of the airplane)
λ	complex roots of stability equation ($u \pm iv$)
μ	airplane density ratio ($m/\rho Sb$)
μ_r	rudder density ratio ($m_r/\rho S_r c_r$)
ρ	density of air
ψ	angle of yaw, radians
$\bar{\psi}$	amplitude of yaw oscillation

METHOD OF ANALYSIS

The only motions considered in the analysis are a yawing of the airplane about its center of gravity and a rotation of the rudder about its hinge. It was shown in reference 3 that the oscillatory stability of an airplane with these two degrees of freedom is essentially the same as when the additional degrees of freedom - that is, rolling and lateral motion - are taken into account. The equations of motion for two degrees of freedom are developed in appendix A. These equations can be obtained from the equations given in reference 1, which include the effect of lateral motion, by making the angle of sideslip equal and opposite to the angle of yaw.

The solution of the equations shows that the motion, in most cases, consists of two superimposed oscillations: one of longer period involving a sensible coupling between yawing of the airplane and swinging of the rudder and the other of shorter period, which corresponds to the oscillation of the rudder when the airplane is acting as a rigid support. The longer-period oscillation has the lower damping and is therefore the one of interest. The period and damping of this oscillation, when given as the distance traveled along the flight path expressed in terms of some characteristic length of the airplane, are independent of speed and, provided the density parameter μ is constant, of airplane size and weight.

On the basis of the equations for two degrees of freedom, the oscillatory stability depends on the following factors:

Airplane and rudder mass characteristics as expressed by

$\mu k_Z{}^2$ airplane moment of inertia

$\mu_r k_r{}^2$ rudder moment of inertia

$\mu_r x_r l$ rudder product of inertia

Yawing-moment characteristics of airplane as expressed by

C_{n_ψ} weathercock stability

$C_{n_{D\psi}}$ damping in yawing

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Yawing-moment characteristics of rudder as expressed by

C_{n_δ} rudder effectiveness

$C_{n_{D\delta}}$ yawing-moment variation with angular velocity of rudder

Rudder hinge-moment characteristics as expressed by

C_{h_ψ} floating-moment parameter

$C_{h_{D\psi}}$ hinge-moment variation with yawing

C_{h_δ} restoring-moment parameter

$C_{h_{D\delta}}$ rudder damping parameter

Inasmuch as only the aerodynamic or viscous damping $C_{h_{D\delta}}$ can be conveniently treated in the equations, it is necessary for the analysis to assume an equivalence between the actual solid friction and a fictitious viscous friction. This equivalence is chosen in such a way that the energy consumed by the solid friction is equal to that consumed by the viscous friction during each cycle. The method of dealing with this equivalence is detailed and discussed in appendix B. The error involved is such that the equations do not show the small irregularities in the motion that will actually result from the presence of the friction. The periods, amplitudes, and conditions for stability, which depend on averaged values, should, however, be reproduced accurately enough.

The study of the effect of the different factors on the rudder-free motion of the airplane was made by a series of computations for the "average" airplane of reference 1 in which the variation of the period and the damping of the lateral oscillation with C_{h_δ} and C_{h_ψ} was determined for various representative values of the other parameters. The basic or average values of the parameters are given as follows:

μk_Z^2	0.926	C_{n_ψ}	-0.064
$\mu_r k_r^2$	0.0222	$C_{n_{D\psi}}$	-0.097
$\mu_r x_r$	0.0	C_{n_δ}	-0.076
l	0.918	$C_{n_{D\delta}}$	-0.0053
$C_{h_{D\psi}}$	$0.918\ C_{h_\psi}$	$C_{h_{D\delta}}$	-0.11

RESULTS AND DISCUSSION

In general, the equations of motion show that, with a rudder having a positive floating tendency, restriction or damping of the rudder movements introduces a lag in the rudder motion that reduces the damping of the lateral oscillation and that may result in continuous or unstable oscillations. If the rudder damping is due to solid friction, the phase lag decreases with an increase in amplitude and the oscillations are continuous and stable; that is, the oscillations are limited to a definite amplitude which depends on the friction. Aerodynamic or viscous damping of the rudder, however, causes a phase lag that does not change with amplitude; hence, if this lag is sufficient, the oscillations will be unstable - that is, will increase indefinitely.

Increasing oscillations due to aerodynamic damping of the rudder.- In figure 1 the damping and the frequency of the oscillation as represented by values of u and v are shown as functions of the floating-moment and restoring-moment parameters of the rudder.

The values shown for u and v are related to the damping and the period of the lateral oscillation by the equations

$$P = 6.28/v$$

$$T_{\frac{1}{2}} = -0.69/u$$

where the period P is in terms of the number of semispan lengths that the airplane moves for a complete cycle and the damping $T_{\frac{1}{2}}$ refers to the number of semispans the airplane moves before the oscillation is damped to one-half its original amplitude.

In figure 1 the control system is assumed to be frictionless. For an average value of the airplane radius of gyration $\left(k_Z = \frac{1}{6}\ b\right)$ the density ratio employed in figure 1 corresponds to a wing loading of 25 pounds per square foot for an airplane of 40-foot span at sea level. A positive value of C_{h_ψ} corresponds to a stabilizing floating tendency, and a negative value of C_{h_δ} corresponds to a stabilizing restoring moment. The magnitude of C_{h_δ} is a

measure of the control forces required to deflect the rudder at zero yaw; a C_{h_δ} of -0.4, for example, corresponds to 150 pounds of pedal force for full deflection of a rudder having an area of 25 square feet and a chord of 3 feet at an indicated speed of 100 miles per hour.

The oscillation becomes undamped for only positive values of C_{h_ψ} and a high degree of aerodynamic balance corresponding to small numerical values of C_{h_δ}. For values of C_{h_δ} numerically greater than a certain magnitude (in this case about 0.12) the damping of the oscillation increases with positive floating tendency as indicated by the curved solid lines. The frequency of the oscillation increases rapidly as C_{h_ψ} is increased, as shown by the dotted lines. The straight line in the lower quadrant is the line of zero weathercock stability with rudder free.

Effect of rudder inertia.- Figure 2 shows the effect of rudder inertia on the oscillatory stability boundary. The effect of rudder inertia on the oscillations is destabilizing but is not very great for reasonable amounts of inertia, as was also shown in reference 2. For this reason and for simpler calculations, rudder inertia has been neglected in most of the subsequent calculations.

Effect of mass balance of rudder.- Mass balancing the rudder has a stabilizing effect on the oscillations, as shown in figure 3. It should be noted that complete mass balance, $\mu_r x_r = 0$, is necessary to provide stability at $C_{h_\psi} = C_{h_\delta} = 0$. Mass overbalance, $\mu_r x_r$ negative, is desirable to insure a margin of stability with complete aerodynamic balance.

Effect of viscous friction in the control system.- The destabilizing effect of viscous damping of the rudder for positive floating tendency is shown for two values of the airplane inertia in figures 4 and 5. Boundaries for increasing oscillations are drawn for arbitrary increases in the value of rudder-damping derivative. In the figures the dotted line drawn tangent to these boundaries determines a region where the oscillations are stable no matter how large the viscous friction in the rudder system is. This line may be called the boundary for complete damping. As the line passes through or very close to the origin, it

corresponds to a fixed value of the ratio of C_{h_ψ} to C_{h_δ}, or the floating ratio of the control surface. For values of the floating ratio numerically greater than the critical value corresponding to the dotted lines of figures 4 and 5 - that is, for points above these lines - the damping may be said to be incomplete because unstable oscillations may occur if the rudder damping is great enough. Similarly, for points on figures 4 and 5 below these lines, the damping is said to be complete because, no matter how great the viscous damping of the rudder, the oscillations will decay.

When the viscous friction increases beyond a certain value that depends on the value of C_{h_ψ}, a further increase has a stabilizing effect. This fact is also shown in figure 6, where C_{h_δ} and $C_{h_{D\delta}}$ are considered as variables and the boundaries for increasing oscillations are drawn for two values of C_{h_ψ}. The maximums on the curves correspond to the boundary for complete damping in figure 5. The value of the aerodynamic damping derivative for the rudder is -0.11 and is indicated by the vertical line. This value is the minimum amount of rudder damping possible. Any larger value would, of course, be due to viscous damping in the rudder control system, such as might be supplied by a dashpot.

The frequency ν of the undamped oscillation for points on the boundaries of figure 6 is shown in figure 7. The angle ϵ by which the rudder lags behind the yaw motion and the relative amplitudes of rudder and yaw δ/Ψ are plotted on the same figure. A comparison of figures 6 and 7 indicates that the phase angle corresponding to the point where the effect of rudder damping is reversed - that is, the minimums of figure 6 - is 45° in both cases. These two figures are very useful in calculating the amplitudes of the undamped oscillations built up when solid friction is present.

Steady oscillations produced by solid friction.- If the effect of viscous friction is destabilizing, as shown in figures 5 and 6, the presence of solid friction will, under certain conditions, result in steadily maintained oscillations. This fact can be shown by using the concept of equivalent viscous friction, which gives the following relation between effective increase of viscous damping, amount of solid friction, and amplitude and frequency of the oscillation:

$$C_{h_{D\delta_f}} = \frac{-4C_{h_f}}{\delta\pi v}$$

This formula is derived in reference 4.

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By use of this relation in conjunction with figure 6, the action of friction can be explained. If the initial disturbance δ/C_{h_f} is very small, the value of effective $C_{h_{D\delta}}$ is, according to the preceding expression, very large and the point representing this value of $C_{h_{D\delta}}$ will lie to the left of the appropriate curve of figure 6. Because this point is in the stable region, the oscillation will damp out completely. If the initial value of $\bar{\delta}/C_{h_f}$ is high enough to place the point on the concave side of the appropriate curve in figure 6, the motion will be unstable and the amplitude will increase. This increase in amplitude decreases the numerical value of the effective $C_{h_{D\delta}}$ until the point on figure 6 moves to the right branch of the curve. Any further increase in amplitude is impossible because it would bring the point on figure 6 into the stable region. If the initial value of $\bar{\delta}/C_{h_f}$ is very large, the effective value of $C_{h_{D\delta}}$ is numerically very small and the point representing it on figure 6 will be to the right of the curve, in the stable region. The amplitude will then decrease and cause the value of $C_{h_{D\delta}}$ to increase until it equals the value at the right branch of the curve.

In figure 8 the amplitudes corresponding to both branches of the curves of figure 6 are plotted against the restoring-moment parameter for two values of the floating-moment parameter. As the condition of aerodynamic balance is approached, the magnitude of the oscillations increases markedly. When a condition is reached at which the oscillations would increase without solid friction, they will be unstable with friction if the initial disturbance is greater than that corresponding to the left branch of the curves of figure 6.

The region where steady oscillations can occur is bounded on one side by the boundary for increasing oscillations without solid friction and on the other by the boundary for complete damping. The variation of the amplitudes of rudder and yaw oscillations in this (shaded) region is shown in figures 9 and 10.

The amplitudes of the steady oscillation are proportional to the frictional hinge-moment coefficient, as shown in appendix B. These amplitudes are therefore directly proportional to the amount of friction and inversely proportional to the square of the indicated speed. Over most of the region the amplitude is extremely small, even with relatively large amounts of friction. On a typical airplane (appendix B) having parameters corresponding to the point shown on figure 9 and with a friction moment of 4 foot-pounds, the maximum amplitude of yawing oscillation occurring when the rudder is freed at 300 miles per hour amounts to less than 0.5°.

Effect of airplane mass characteristics.- As the moment-of-inertia ratio μk_Z^2 of the airplane about the vertical axis is increased, the region where steady oscillations may take place is extended but the boundary for increasing oscillations is unchanged (fig. 11). An increase in moment of inertia is equivalent to an increase in wing loading if the airplane size and mass distribution are held constant. Increased wing loading therefore increases the likelihood of steady oscillations due to friction, but does not alter the conditions for unstable oscillations. Additional calculations not shown here indicate that the region of unstable oscillations is not altered appreciably by variations in μk_Z^2 of from 0.2 to 5.0.

Effect of varying the weathercock stability by changing the vertical tail area.- The effect of varying the weathercock stability of an airplane by changing the area of the vertical tail is shown by figure 12. The effect of changing vertical-tail area on the weathercock stability and other factors is as follows:

S_t/S	C_{n_ψ}	$C_{n_{D\psi}}$	C_{n_δ}	$C_{n_{D\delta}}$	$C_{h_{D\delta}}$
0.04	-0.032	-0.076	-0.051	-0.0036	-0.09
.06	-.064	-.097	-.076	-.0053	-.11
.10	-.128	-.130	-.126	-.0089	-.14

Effect of rudder effectiveness.- Increase in rudder effectiveness, such as would be obtained, for example, by increasing the ratio of movable to fixed tail surface, has an adverse effect on the dynamic stability, as shown in figure 13. The critical floating ratio varies inversely with rudder effectiveness.

Summary chart showing the effect of several parameters on the limiting conditions for steady oscillations.- Figure 14 summarizes the results of figures 11 to 13 and shows the effects of four significant parameters on the region where steady oscillations may take place. The term $d\alpha_e/d\delta$ in the abscissa is directly proportional to C_{n_δ} (reference 1). Additional values of C_{n_ψ} and μk_Z^2 are included to cover the practical range of weathercock stability and most of the range of moment-of-inertia ratio for present-day airplanes. The smallest value of μk_Z^2 shown, 0.926, corresponds to that for a wing loading of 25 pounds per square foot at sea level for a pursuit airplane having a radius of gyration about the vertical axis equal to one-sixth the span. If the span of the airplane were increased by a certain factor, the wing loading corresponding to a given μk_Z^2 would increase by the same factor, other conditions remaining constant.

Effect of mass overbalance with solid friction.- It has already been shown (fig. 3) that mass overbalance of the rudder (rudder center of gravity ahead of hinge) has a beneficial effect on the boundary for increasing oscillations. The effect on the boundary for steady oscillations is also beneficial, as shown in figure 15. The dotted line corresponds to a mass-balanced rudder; the full line corresponds to a rudder the center of gravity of which is 10 percent of the rudder chord ahead of the hinge and the mass of which is about 1 percent of the mass of the airplane. This rudder weight is considerably more than usual but could be reduced by increasing the distance between rudder hinge and rudder center of gravity.

CONCLUSIONS

The calculations presented in this paper show the existence of oscillations of constant amplitude in a rudder system having friction and certain hinge-moment characteristics. The charts presented show the conditions that tend to minimize or eliminate these undesirable oscillations and are intended as a guide to the design of airplanes having rudders with a stabilizing floating tendency. The results of these calculations indicate the following conclusions:

1. A closely balanced rudder having too great a positive floating tendency will be dynamically unstable if the control is freed.

2. Under conditions of dynamic stability for a rudder with a positive floating tendency, a continuous oscillation of fixed amplitude may be caused by friction in the control system.

3. The amplitude of the steady oscillation is proportional to the amount of friction and, for all practical purposes, the oscillation may be eliminated by reducing the friction, provided the aerodynamic balance is not too nearly complete.

4. The amplitude of the steady oscillation is inversely proportional to the square of the indicated speed.

5. The steady oscillation can be eliminated by using a sufficiently small floating ratio or by mass overbalance of the rudder.

6. A positive floating tendency can be used to compensate for a lack of weathercock stability if the control system is designed for small friction.

Flight tests will be necessary to indicate the maximum amount of steady oscillation that is allowable on an airplane.

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APPENDIX A

EQUATIONS OF MOTION FOR THE CASE OF VISCOUS FRICTION IN THE RUDDER CONTROL SYSTEM

It was shown in reference 3 that the lateral motion of the center of gravity and the rolling motion may be neglected in the analysis of the lateral oscillations with

free rudder. The number of degrees of freedom is thereby reduced to two: namely, angle of yaw and rudder deflection.

The equations of motion are:

$$(2\mu k_Z^2 D^2 - C_{n_{D\psi}} D - C_{n_\psi})\psi + (-C_{n_{D\delta}} D - C_{n_\delta})\delta = 0$$

$$[(2\mu_r k_r^2 + 2\mu_r x_r l)D^2 - C_{h_{D\psi}} D - C_{h_\psi}]\psi + (2\mu_r k_r^2 D^2 - C_{h_{D\delta}} D - C_{h_\delta})\delta = 0$$

Substituting $\psi = Me^{\lambda s}$ and $\delta = Ne^{\lambda s}$ in these equations indicates that λ must be a root of the fourth-degree equation

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + E\lambda + F = 0 \qquad (1)$$

where

$$A = 4\mu k_Z^2 \mu_r k_r^2$$

$$B = -2\mu k_Z^2 C_{h_{D\delta}} + 2(C_{n_{D\delta}} - C_{n_{D\psi}})\mu_r k_r^2 + 2\mu_r x_r l C_{n_{D\delta}}$$

$$C = -2\mu k_Z^2 C_{h_\delta} + C_{n_{D\psi}} C_{h_{D\delta}} - C_{h_{D\psi}} C_{n_{D\delta}} + 2\mu_r k_r^2 (C_{n_\delta} - C_{n_\psi}) + 2\mu_r x_r l C_{n_\delta}$$

$$E = C_{n_{D\psi}} C_{h_\delta} - C_{h_{D\psi}} C_{n_\delta} - C_{h_\psi} C_{n_{D\delta}} + C_{n_\psi} C_{h_{D\delta}}$$

$$F = C_{n_\psi} C_{h_\delta} - C_{h_\psi} C_{n_\delta}$$

The boundary for divergence is obtained by setting $F = 0$ and that for increasing oscillations is found by setting Routh's discriminant

$$R = BCE - AE^2 - FB^2 = 0 \qquad (2)$$

The roots of equation (1) can be easily found when equation (2) is satisfied; in this case the values of λ corresponding to the undamped oscillation are

$$\lambda = \pm i\nu = \pm i\sqrt{E/B}$$

The amplitude ratio and phase difference between rudder and yaw for the undamped oscillation can be found by substituting $i\nu$ for λ in the expression

$$\frac{2\mu k_Z^2\lambda^2 - C_{n_{D\psi}}\lambda - C_{n_\psi}}{C_{n_{D\delta}}\lambda + C_{n_\delta}}$$

which may be written as $p + iq$. Then the amplitude ratio $\bar{\delta}/\bar{\psi} = \sqrt{p^2 + q^2}$ and the angle of lag ϵ of δ behind ψ is equal to $-\tan^{-1}\frac{q}{p}$.

If C_{h_δ} and C_{h_ψ} are considered as variables and all other parameters held constant except $C_{h_{D\psi}}$, which is proportional to C_{h_ψ}, curves of the type in figures 1 to 3 result from the relations $R = 0$ and $F = 0$. If the rudder moment of inertia is neglected, considerable simplification in the expression for R results. Equation (2) then reduces to

$$R = CE - FB = 0$$

If $C_{h_{D\delta}}$ is considered as a variable, in addition to C_{h_δ} and C_{h_ψ}, a family of curves can be drawn, as on figures 4 and 5. The envelope of the curves in these figures can be found by solving simultaneously the equations

$$R = CE - FB = 0$$

and

$$\frac{\partial R}{\partial C_{h_{D\delta}}} = 0$$

The result is a relation between C_{h_δ} and C_{h_ψ}. The straight lines giving the boundary for complete damping on figures 4 and 5 were obtained in this way. This boundary determines the region where an increase of the viscous damping parameter $-C_{h_{D\delta}}$ can cause dynamic instability and is of significance in determining the effect of solid friction.

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APPENDIX B

TREATMENT OF SOLID FRICTION IN CONTROL SYSTEM

Approximate Method of Calculating Amplitudes of Steady Oscillations

Previous work (reference 5) has shown that certain dynamical systems can, in the presence of solid friction, build up constant-amplitude oscillations that would not exist in the absence of friction. This work, however, was limited to the case of continuous motion of the rudder - that is, motion in which the rudder does not stop moving during each cycle. The effect of friction in the case of discontinuous motion has been discussed in the previously mentioned document by Schairer and Bush of Boeing Aircraft Co. The main results of their analysis agree, in general, with this report but do not include as many factors and do not agree quantitatively with the present work.

It is shown in the body of the report by approximating the solid friction by an equivalent viscous friction that, if viscous friction is destabilizing, solid friction will result in constant-amplitude oscillations. The amplitude of the oscillations is given by

$$\frac{\bar{\delta}}{C_{h_f}} = -\frac{4}{\pi v C_{h_{D\delta_f}}}$$

where $C_{h_{D\delta_f}}$ is the value of the viscous damping required to make $R = 0$ minus the value of $C_{h_{D\delta}}$ due to aerodynamic damping of the rudder. This expression for the amplitude in terms of the amount of solid friction, the amount of viscous friction, and the frequency is derived in reference 4.

Numerical Example Using Approximate Method

The calculation of the amplitude of the steady oscillation due to solid friction will now be made for a specific airplane having the following characteristics:

$\mu k_Z{}^2$ 1.852

$\mu_r k_r{}^2$ 0

C_{h_δ} . . . . -0.2

C_{h_ψ} 0.3

The other parameters are the same as those listed in the section on Method of Analysis.

The point representing these values is indicated on figures 9 and 10 by a crossed circle. In order to find the value of $C_{h_{D\delta}}$ for $R = 0$, let $C_{h_{D\delta}} = x$ and calculate the coefficients of equation (1). The following values are obtained:

$A = 0$

$B = -3.704x$

$C = 0.742 - 0.097x$

$E = 0.0419 - 0.064x$

$F = 0.0356$

$R = CE - FB = 0.006204x^2 + 0.0803x + 0.03109 = 0$

Value of	Corresponding to steady oscillation	Corresponding to minimum conditions
x	-0.399	-12.55
B	1.479	46.46
E	.0674	.8447
$v = \sqrt{E/B}$	.2138	.1348
$C_{h_{D\delta_f}}$	-.289	-12.44
$\bar{\delta}/C_{h_f}$	20.6	.76
$\bar{\delta}/\bar{\psi}$	1.4	.18
$\bar{\psi}/C_{h_f}$	14.6	4.2

The values in the last column correspond to the minimum initial disturbance necessary for building up the oscillation to a constant amplitude.

Any disturbance greater than 0.76 radian of rudder angle and 4.2 radians of yaw angle per frictional hinge-moment coefficient will therefore build up to a steady oscillation the amplitudes of which are 20.6 for the rudder and 14.6 for the yaw angle, in the same units.

These nondimensional values can be expressed in physical units as follows: If the rudder dimensions, frictional hinge moment, and indicated speed are

Rudder area, S_r, sq ft	18
Rudder chord, c_r, ft	3
Frictional hinge moment, H_f, ft-lb	4
Airspeed, V, mph	300
Wing span, b, ft	42.4

then

$$C_{h_f} = \frac{H_f}{\frac{1}{2}\rho V^2 S_r c_r} = 0.000322$$

$$\overline{\psi} = 14.6 \times 0.000322 \times 57.3 = 0.26^\circ$$

and

$$\overline{\delta} = 1.4 \times 0.26 = 0.36^\circ$$

for the amplitudes of the steady oscillation, and

$$\overline{\psi} = 4.2 \times \frac{0.26}{7.3} = 0.074^\circ$$

$$\overline{\delta} = 0.76 \times \frac{0.36}{20.6} = 0.014^\circ$$

for the minimum disturbance required to start the oscillation. In this case, it is seen that the steady oscillation, having a maximum amplitude of less than 0.5°, would hardly be perceptible in flight.

The period of the steady oscillations is

$$\frac{2\pi}{0.2138} \times \frac{42.4 \times 60}{2 \times 300 \times 88} = 1.42 \text{ sec}$$

Comparison with More Exact Calculation of the Effect of Solid Friction on the Motion

In order to check the approximate theory, a step-by-step calculation of the rudder motion following certain initial disturbances was made for two conditions. The results of these calculations are shown in figures 16 and 17. Each time the rudder motion stopped the rudder became locked by the friction and the subsequent motion was calculated for that condition until the force on the rudder exceeded the force of friction, when the rudder moved back and another step in the calculations was made. The steps in the calculation are thus of two alternating kinds: rudder-fixed motion and rudder-free motion. The motion of the rudder under these conditions has flat-top peak as shown in the figures and also in flight records.

Figure 16 shows the motion corresponding to the numerical example given in the previous section. The motion without solid friction is shown for comparison. An arbitrarily chosen initial displacement in yaw was taken. The effect of friction in causing the motion to build up is clearly shown. The vertical lines on the right of the figure give the amplitudes as previously calculated by the approximate method.

Figure 17 shows the motion for an airplane having zero weathercock stability for two different disturbances. The motion following the large disturbance consists of damped oscillations; the motion following the small disturbance leads to slightly increasing oscillations. It must be presumed, therefore, that each disturbance will eventually reach the same constant value. The amplitudes calculated by the approximate method are again shown at the right side of the figure.

The agreement between the amplitudes as calculated by the exact and the approximate methods is better in figure 17 than in figure 16. In both cases the approximate calculation gives the higher value. The agreement should be considered good in view of the approximations involved and, in any case, the values given by the approximate method are on the conservative side.

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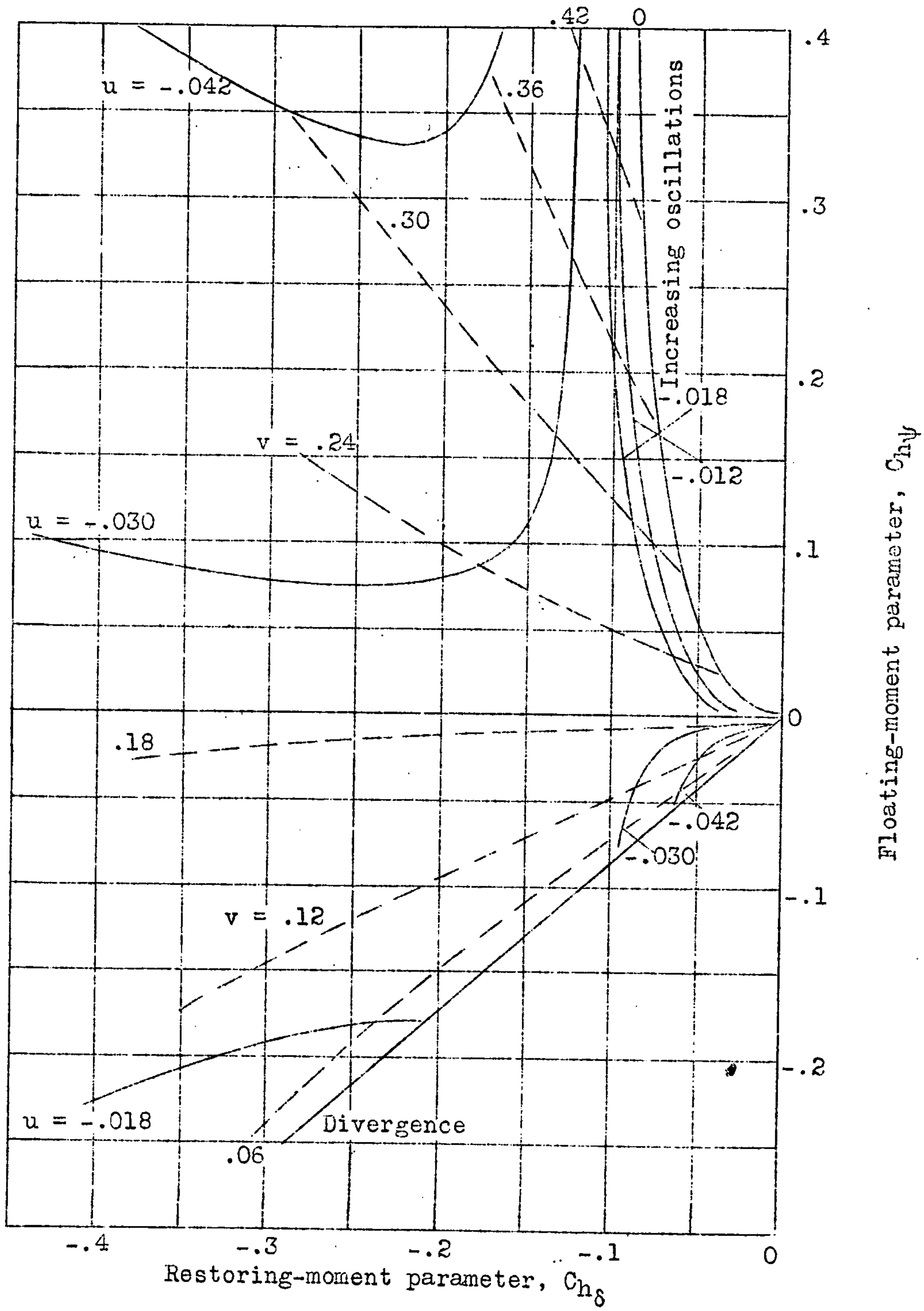

Figure 1.- Curves of constant period and constant damping of yaw-rudder oscillation. $\lambda = u \pm iv$; $\mu k_Z^2 = 0.926$; $C_{n_\psi} = -0.064$; $\mu_r k_r^2 = 0.0222$.

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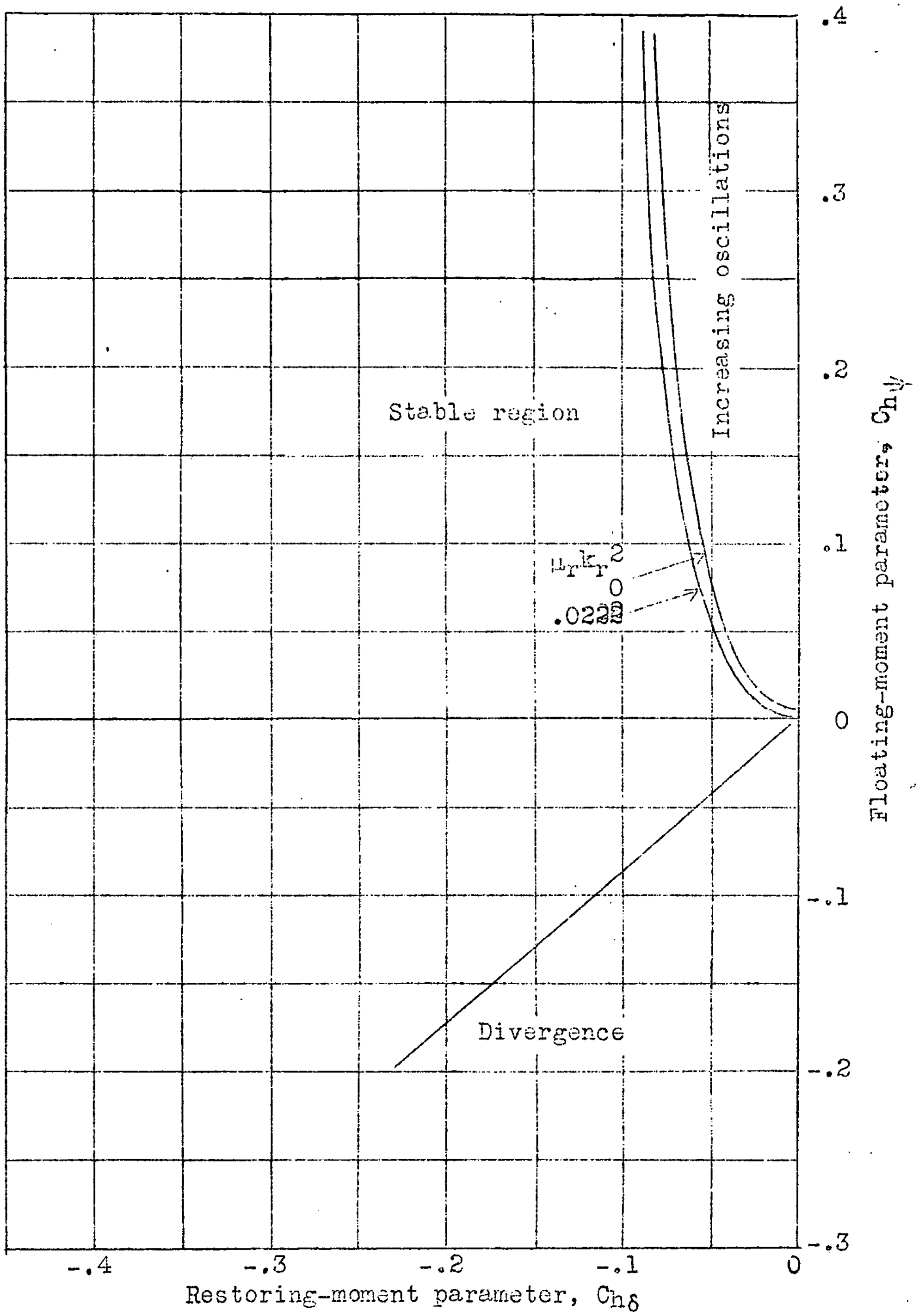

Figure 2.- Effect of rudder inertia on rudder-free stability. $\mu k_Z{}^2 = 0.926$; $C_{n_\psi} = -0.064$.

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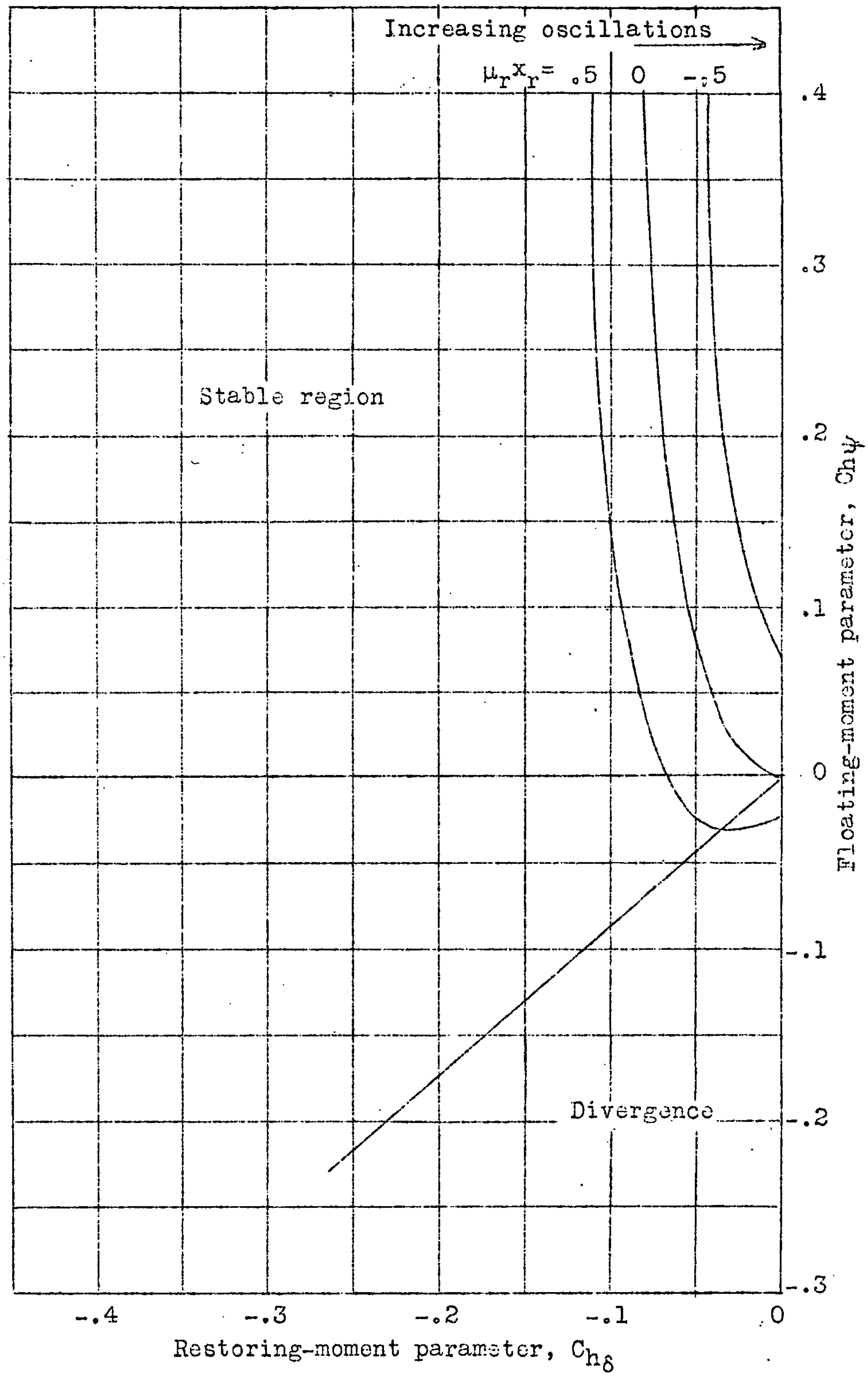

Figure 3.- Effect of mass balance of the rudder on rudder-free stability. $\mu k_Z^2 = 0.926$; $C_{n_\psi} = -0.064$; $\mu_r k_r^2 = 0$.

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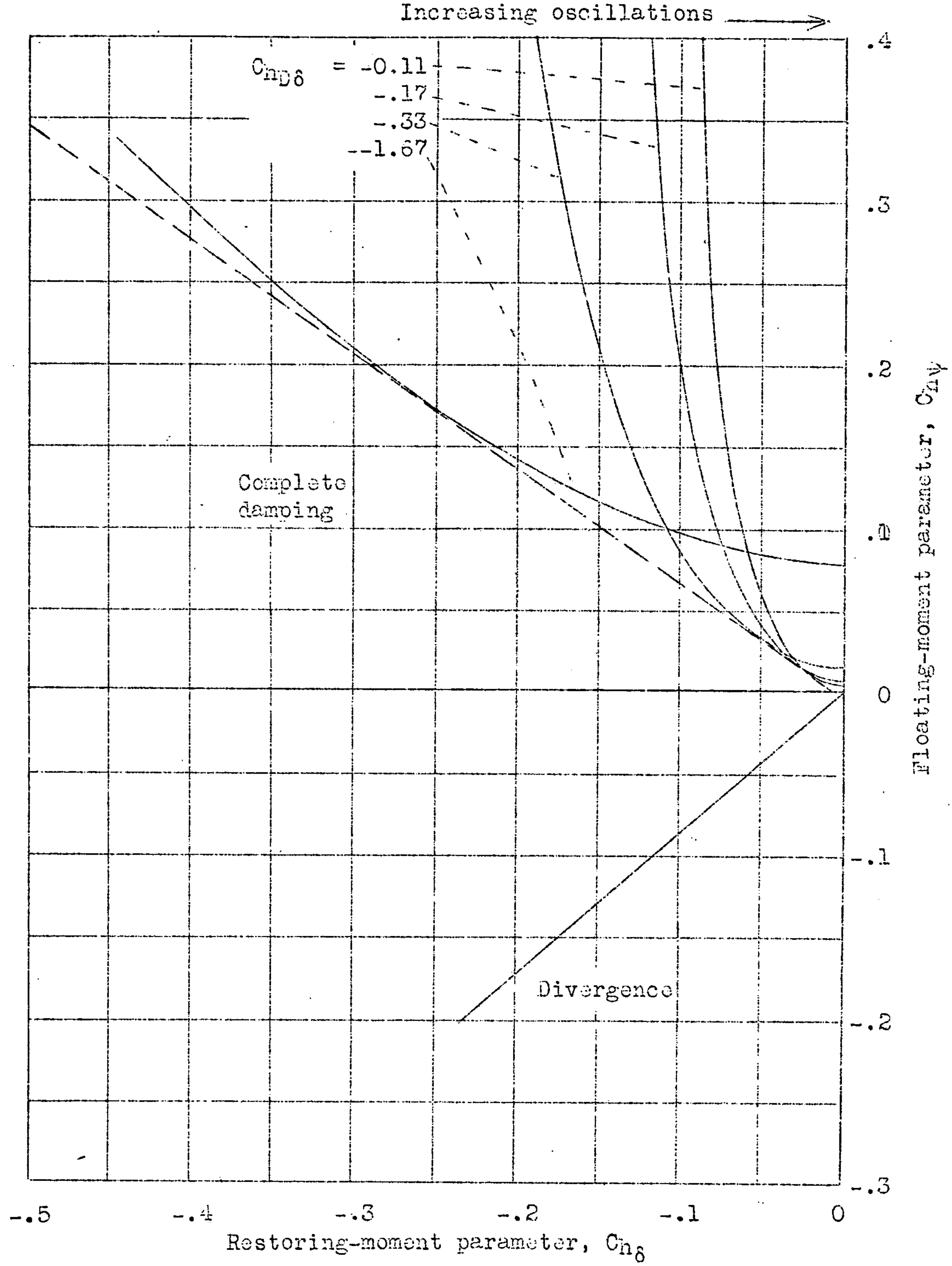

Figure 4.- Effect of rudder damping on boundary for increasing oscillations. $\mu k_Z^2 = 0.926$; $C_{n_\psi} = -0.064$; $\mu_r k_r^2 = 0.0222$.

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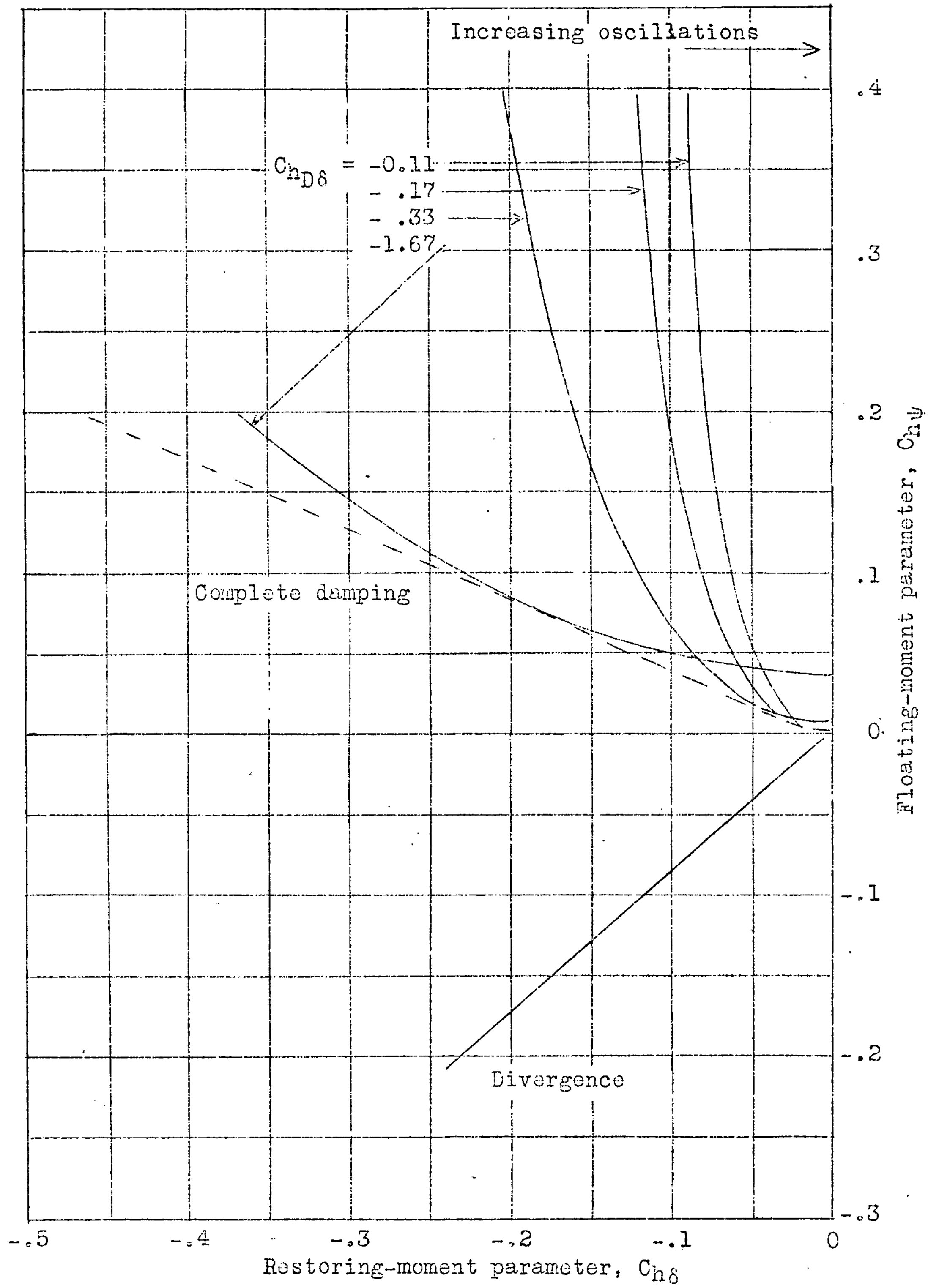


Figure 5.- Effect of rudder damping on boundary for increasing oscillations. $\mu k_Z^2 = 1.852$; $C_{n_\psi} = -0.064$; $\mu_r k_r^2 = 0.0222$.

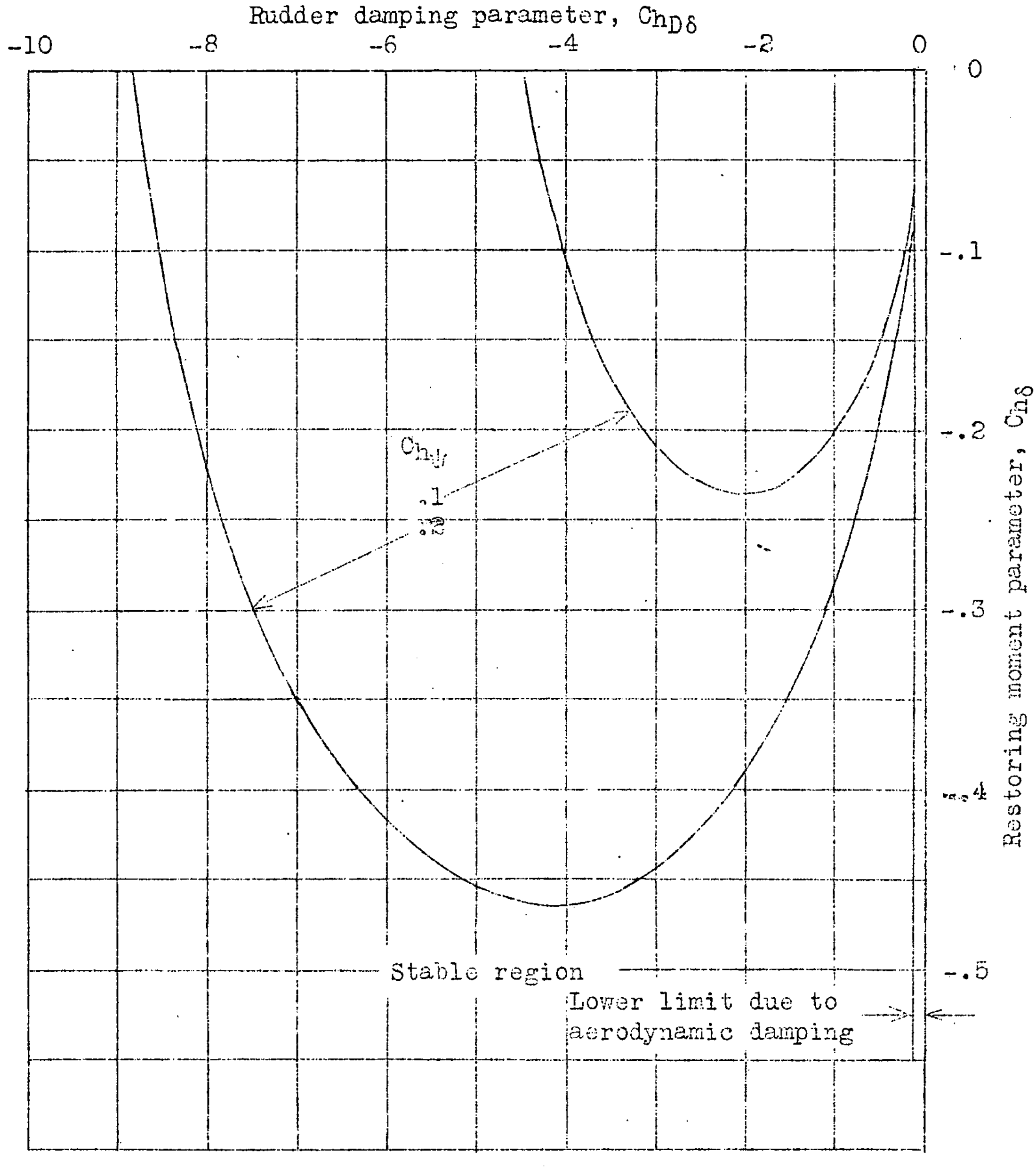
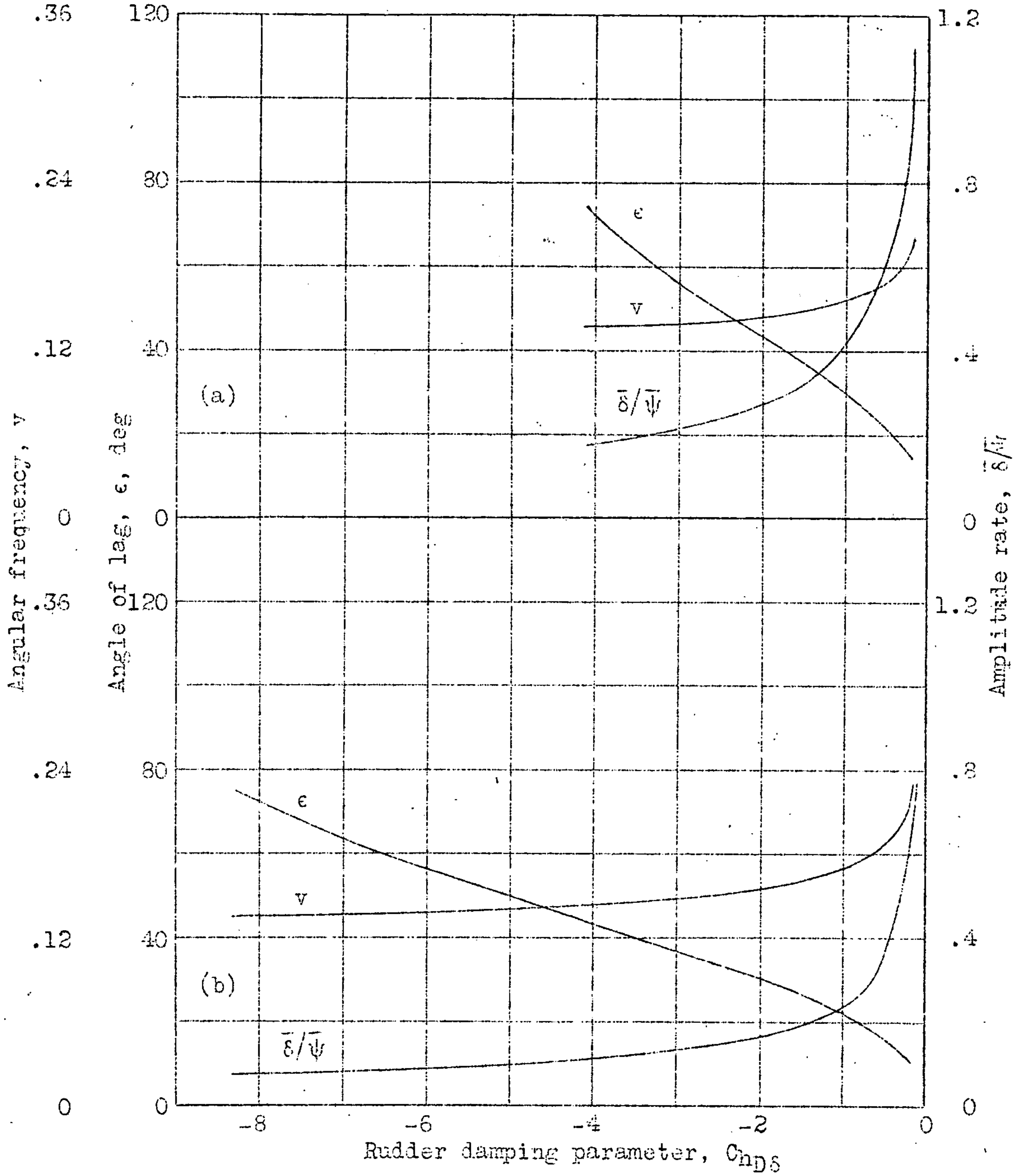


Figure 6.- Relation between C_{h_δ} and $C_{h_{D\delta}}$ for constant values of C_{h_ψ}, for the condition of undamped oscillations. $\mu k_Z^2 = 1.852$; $\mu_r k_r^2 = 0.0222$; $C_{n_\psi} = -0.064$.



(a) $C_{h_{\psi}} = 0.1$ (b) $C_{h_{\psi}} = 0.2$

Figure 7.- Variation of amplitude ratio $\bar{\delta}/\bar{\psi}$, angular frequency v, and angle of lag ε with rudder damping for the condition of undamped oscillations. $\mu k_Z^2 = 1.852$; $C_{n_{\psi}} = -0.064$.

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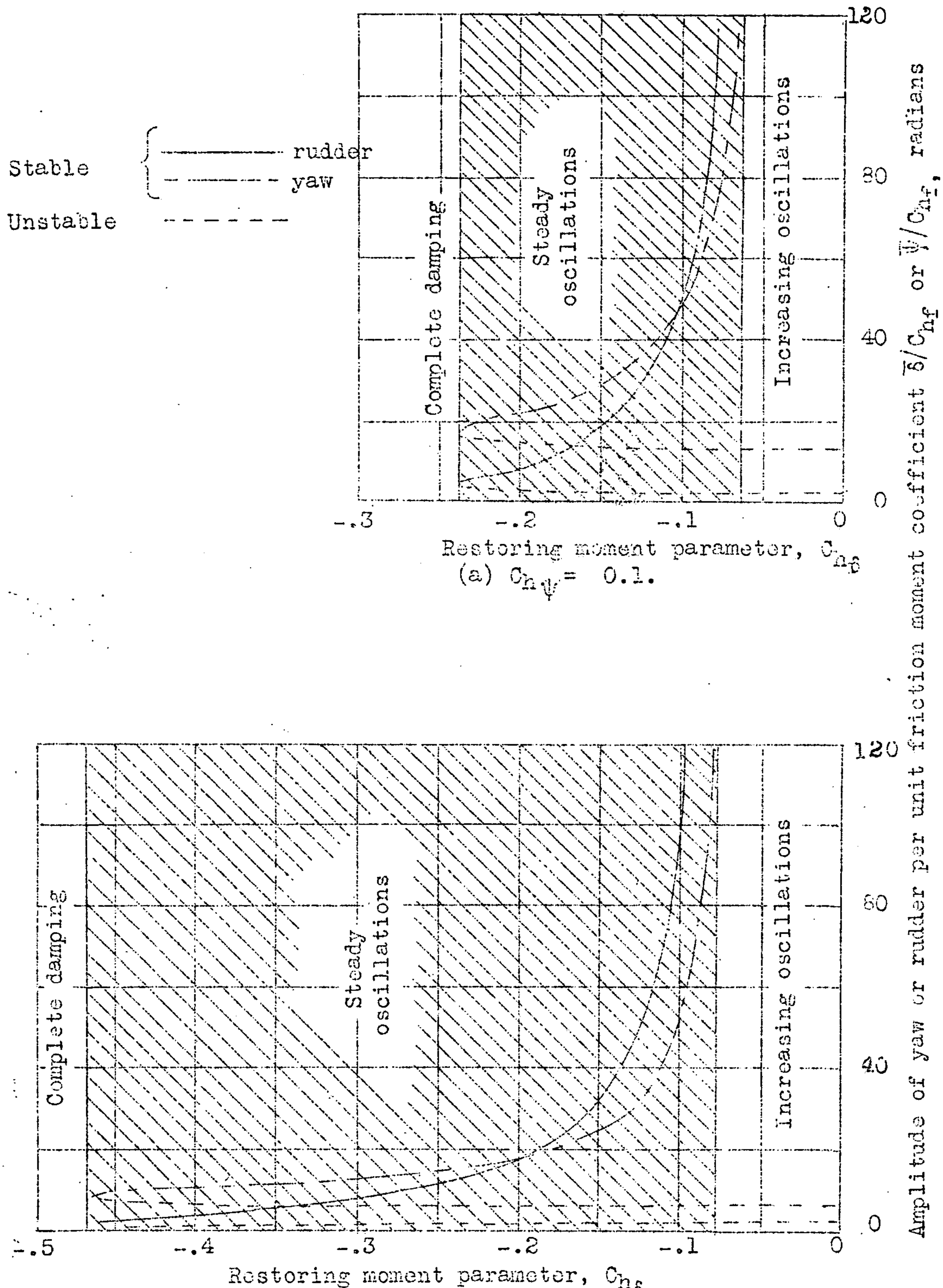

(a) $C_{h_\psi} = 0.1$.

(b) $C_{h_\psi} = 0.2$.

Figure 8.- Variation of amplitude of steady oscillations with aerodynamic balance. $C_{n_\psi} = -0.064$; $\mu k_Z{}^2 = 1.852$.

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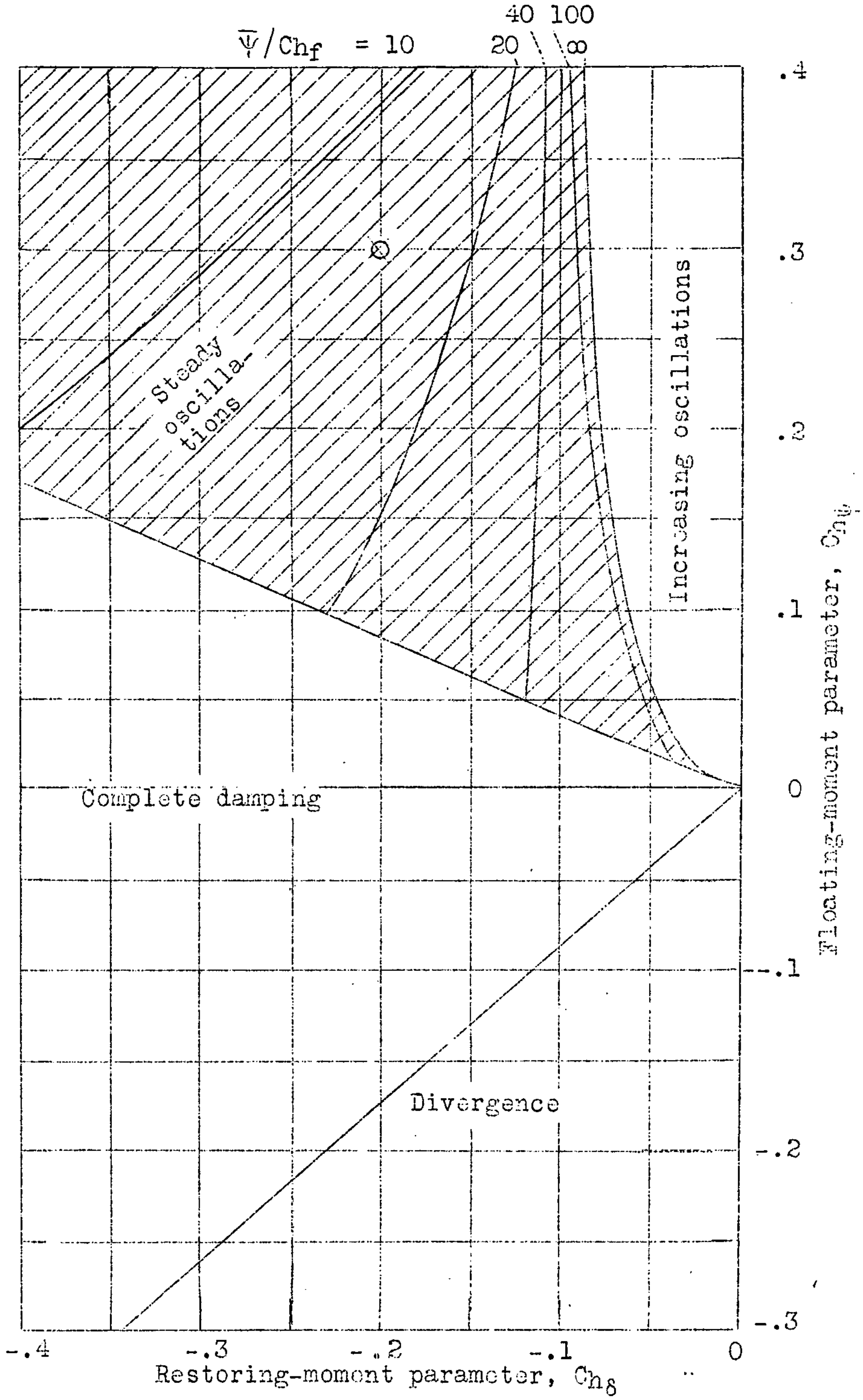

Figure 9.- Variation of yaw amplitude of steady oscillations with aerodynamic balance and floating tendency. $\mu k_Z^2 = 1.852$; $C_{n_\psi} = -0.064$; $\mu_r k_r^2 = 0.0222$.

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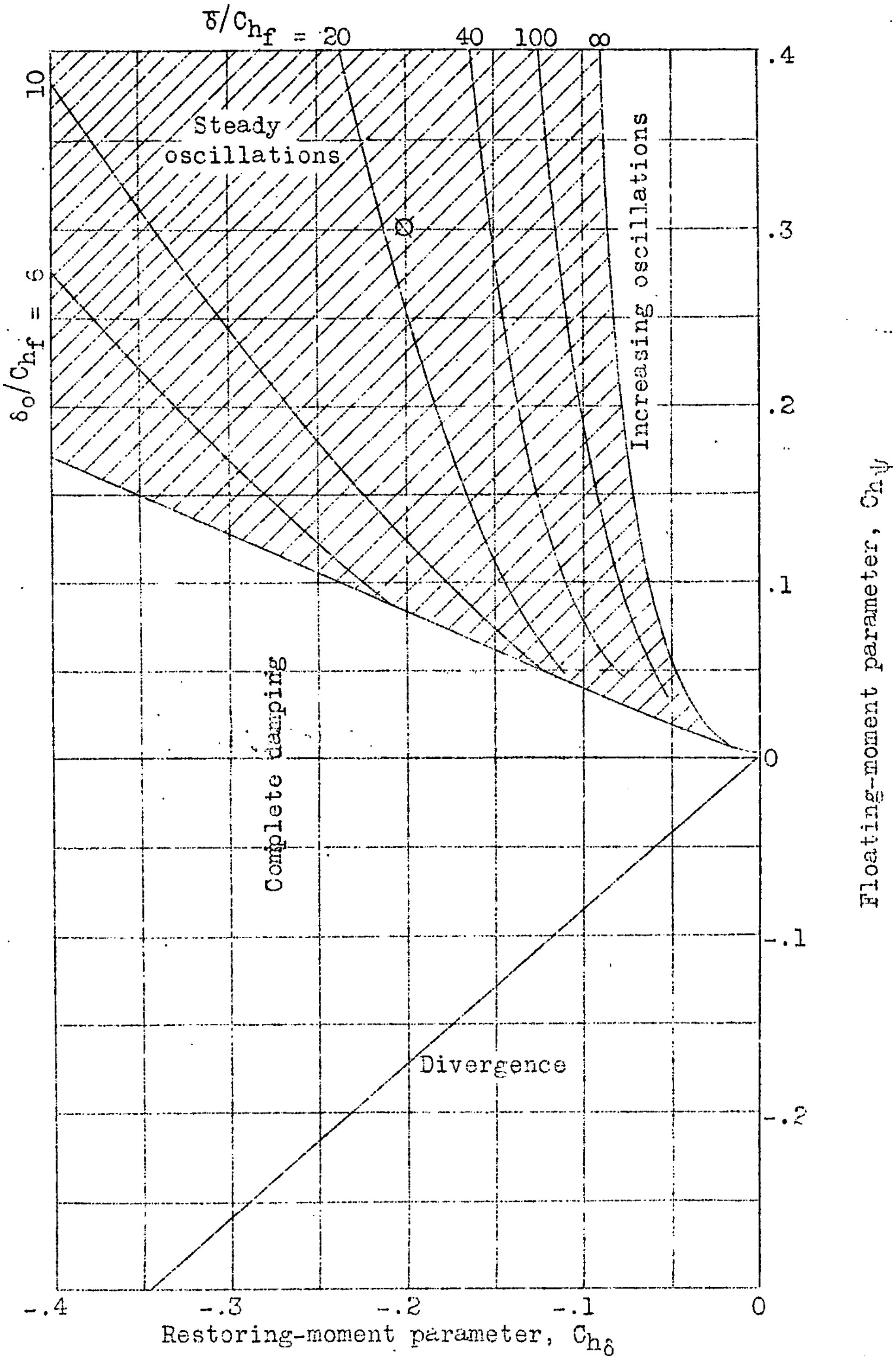

Figure 10.- Variation of rudder amplitude of steady oscillations with aerodynamic balance and floating tendency. μk_z^2=1.852; C_{n_ψ} = -0.064; $\mu_r k_r^2$=0.0222.

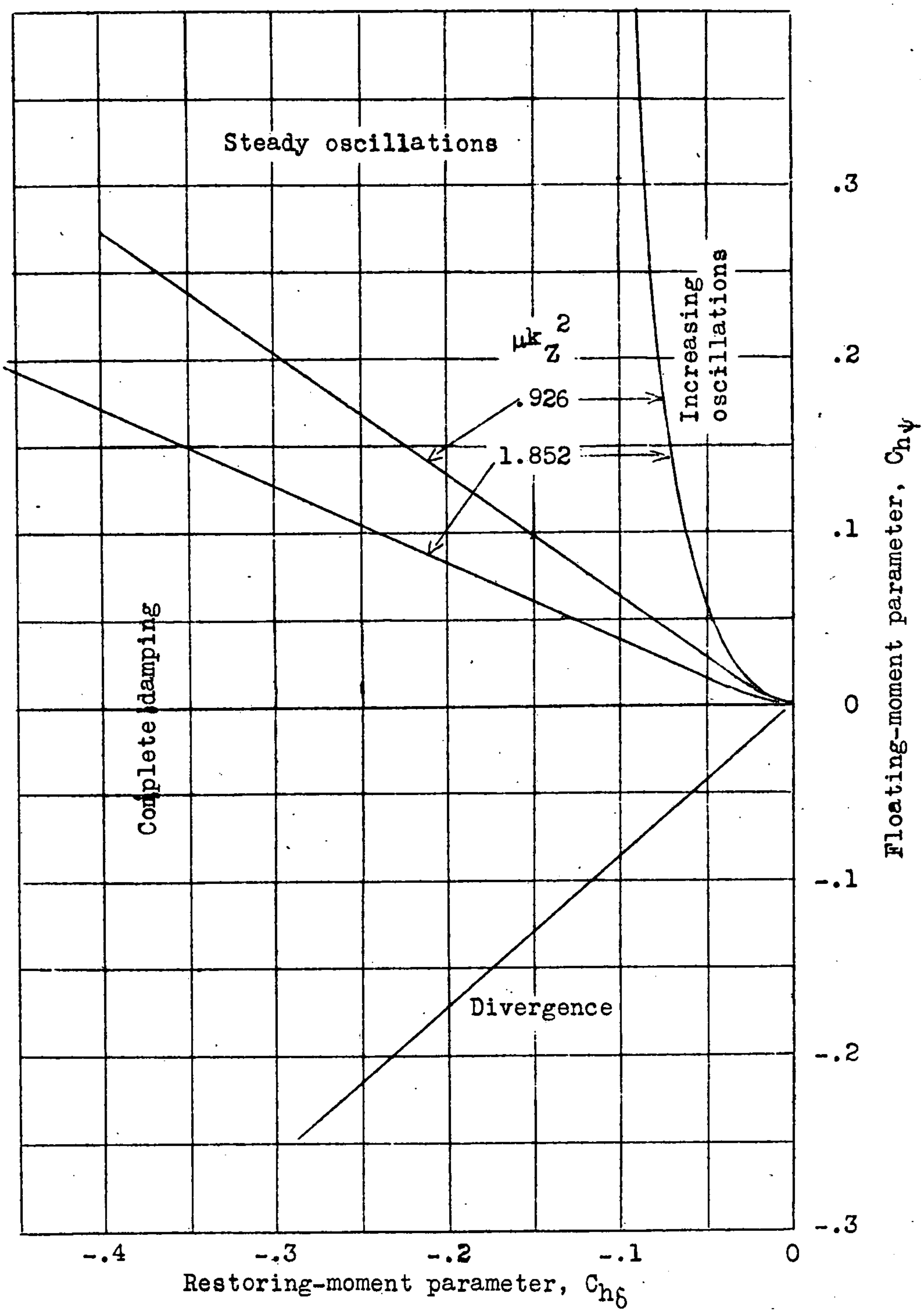

Figure 11.- Effect of airplane density and radius of gyration on dynamic stability. $C_{n_\psi} = -0.064$ $\mu_r k_r^2 = 0.0222$.

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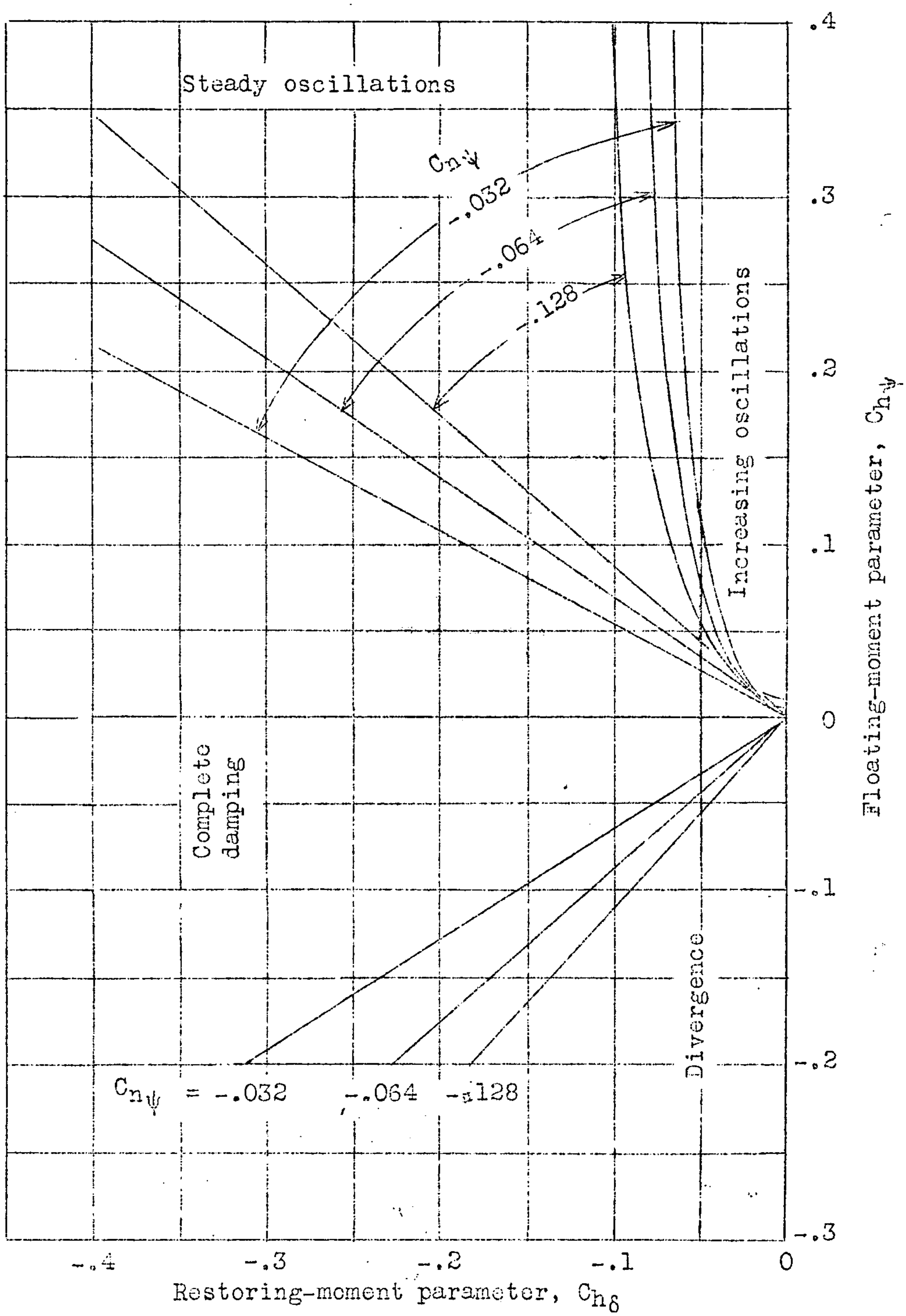

Figure 12.- Effect of weathercock stability on boundaries for increasing oscillations, steady oscillations, and divergence. $\mu k_Z^2 = 0.926$.

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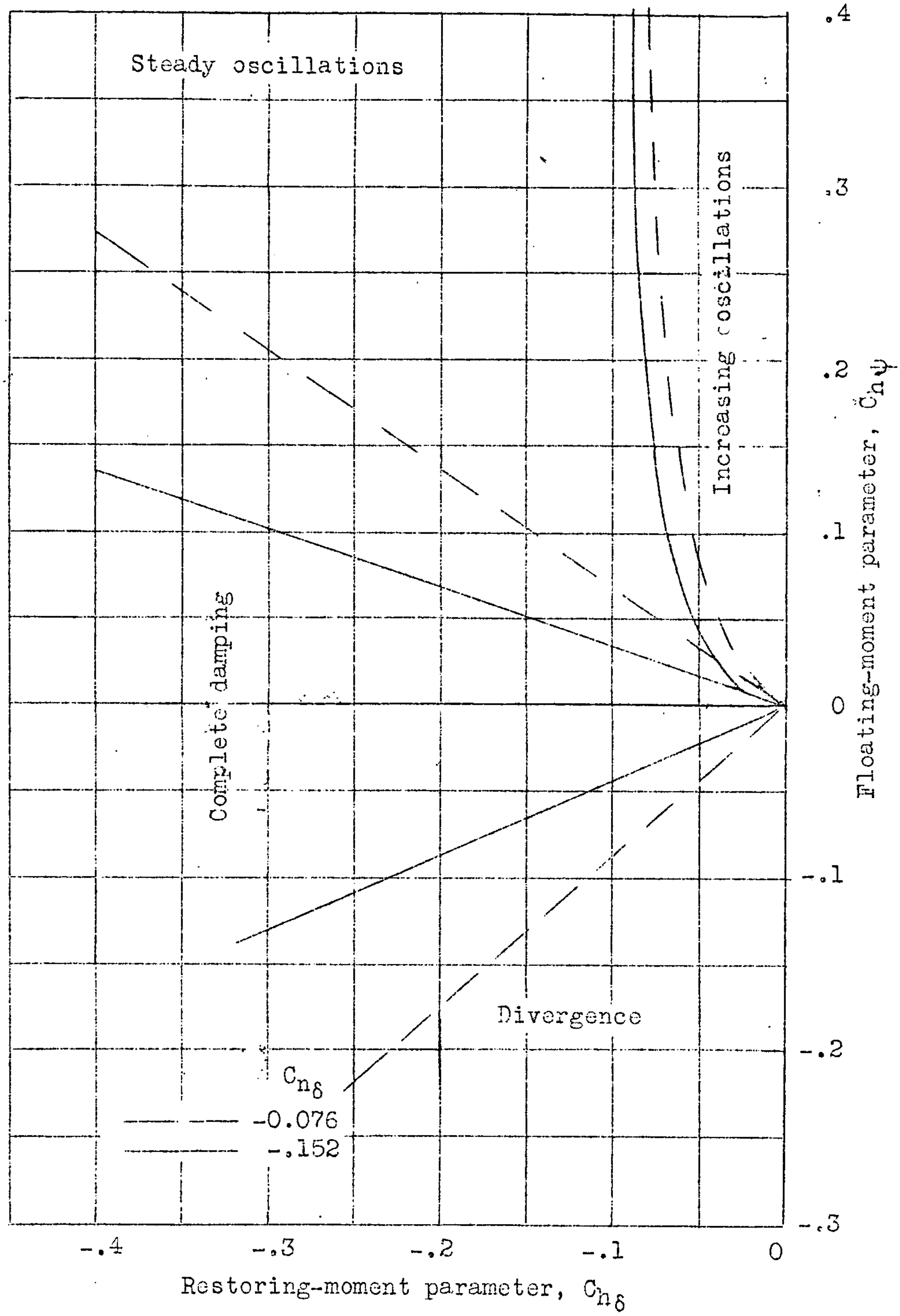

Figure 13.- Effect of rudder effectiveness on boundaries for increasing oscillations, steady oscillations, and divergence. $\mu k_Z^2 = 0.926$.

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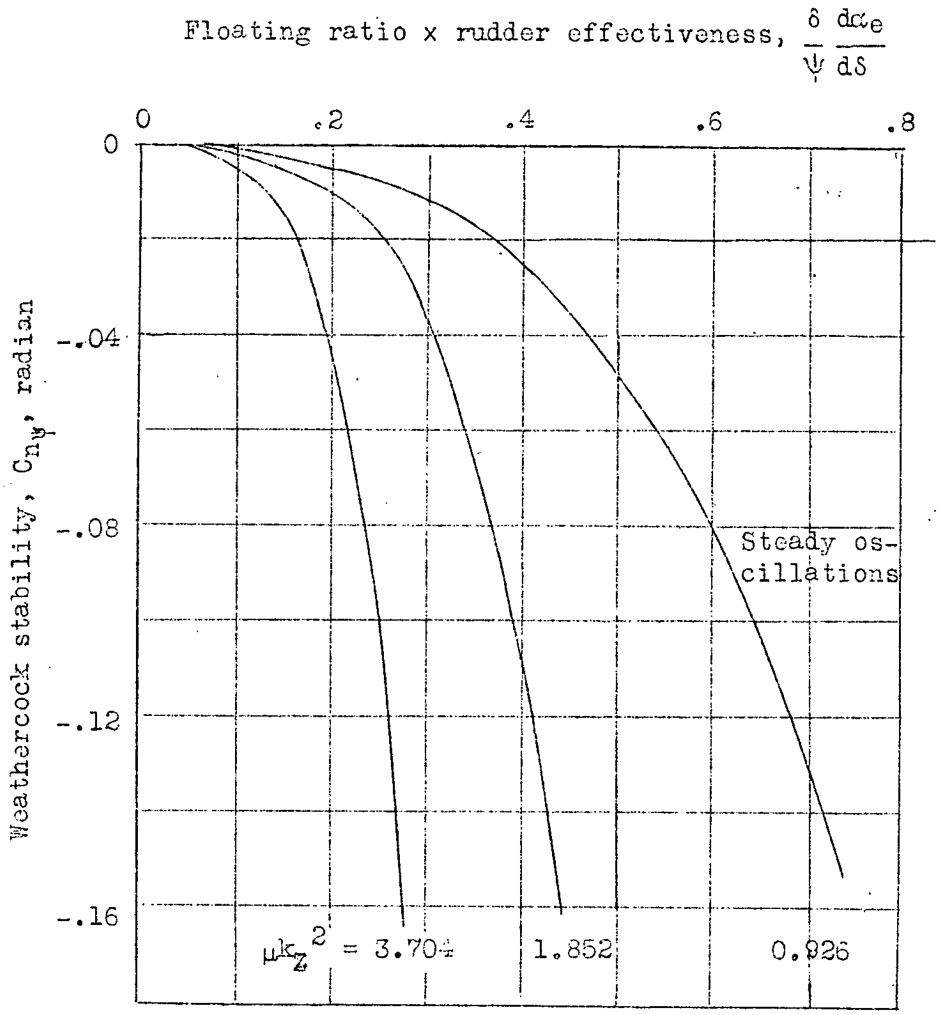

Figure 14.- Limits for steady oscillations in terms of tail size, rudder effectiveness, floating ratio, airplane density and radius of gyration.

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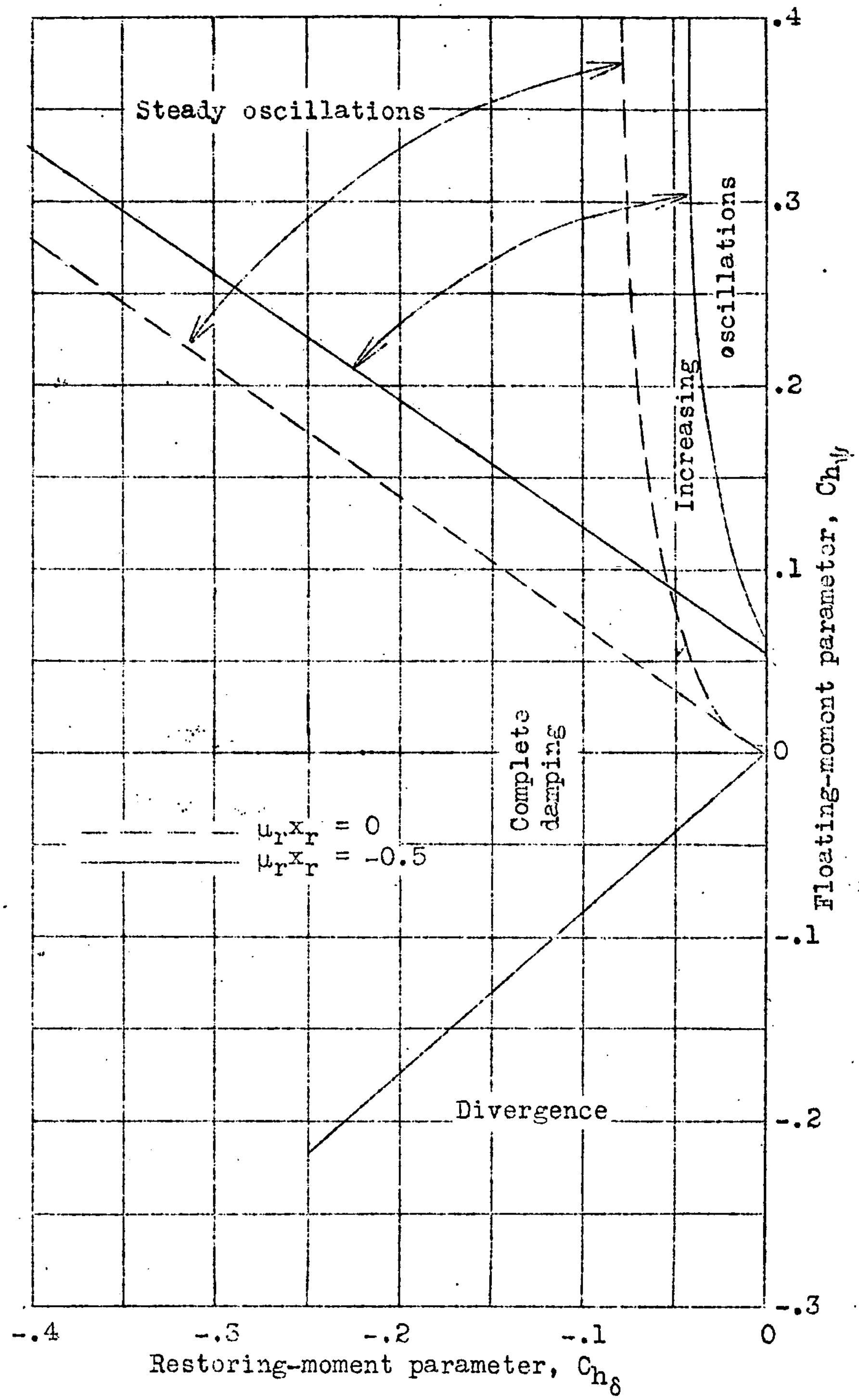

Figure 15.- Effect of mass overbalance of rudder on dynamic stability. $\mu k_Z^2 = 0.926$; $C_{n_\psi} = -0.064$.

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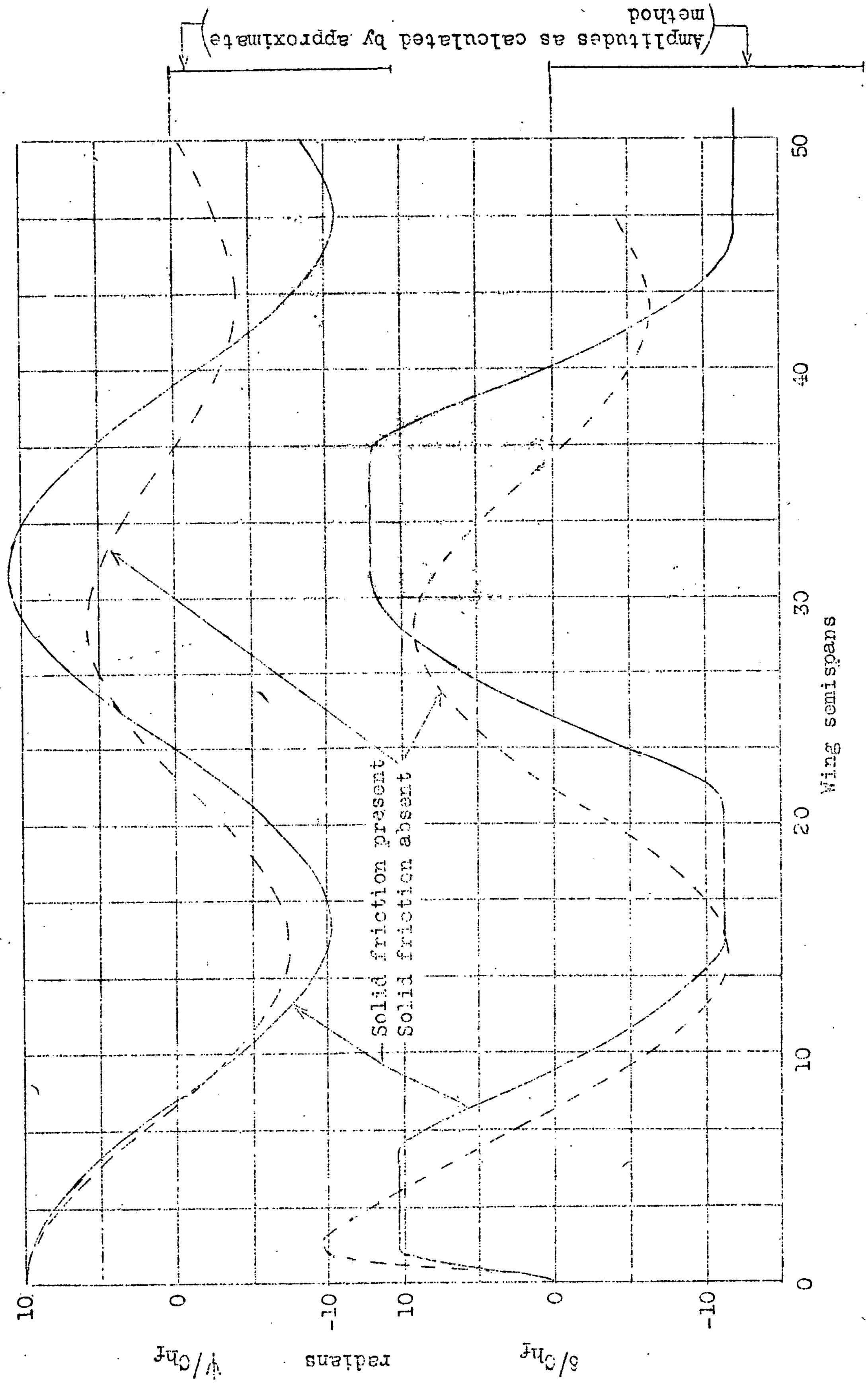

Figure 16.- Motion of rudder and airplane under the influence of solid friction in the rudder control system.

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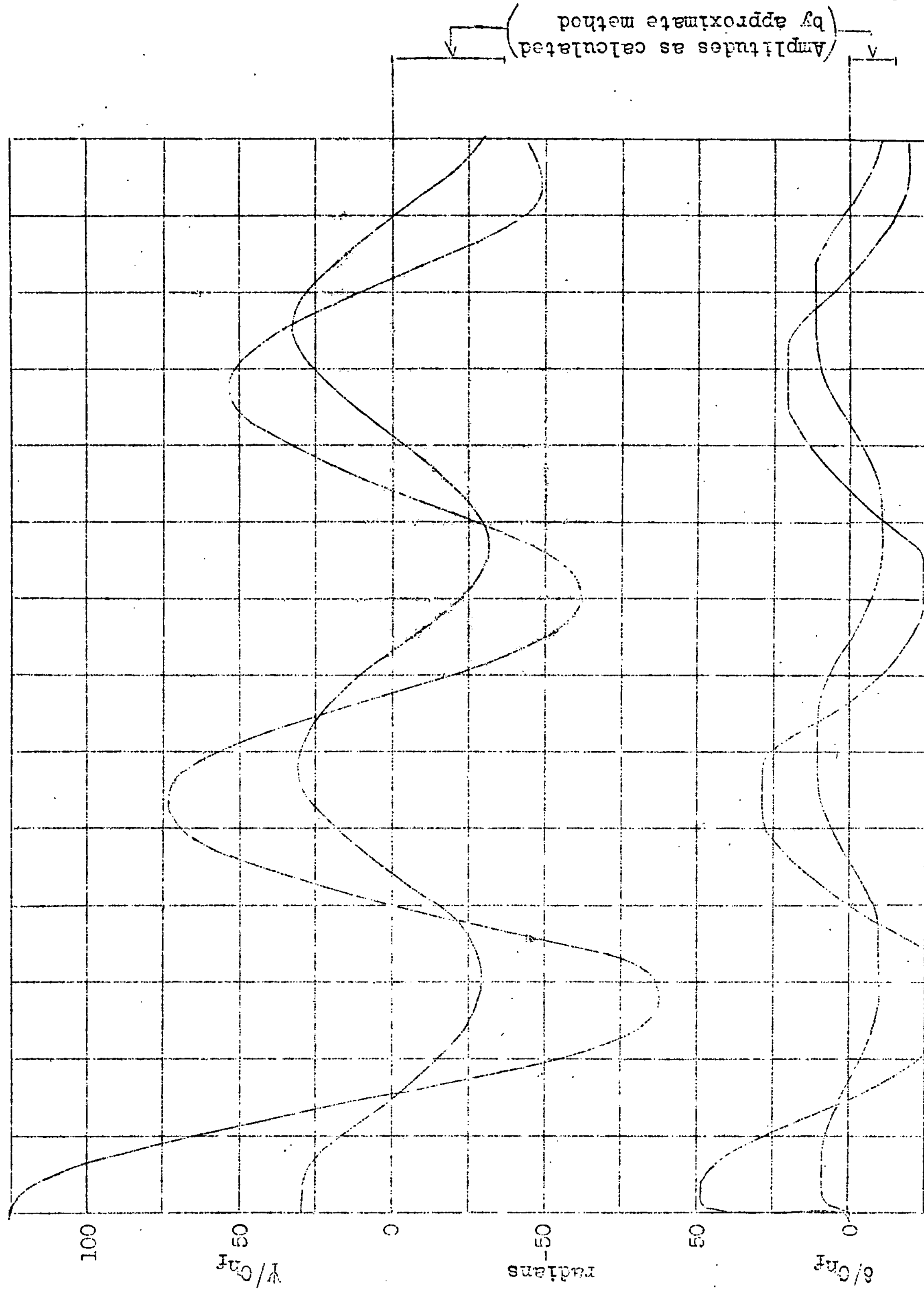

Figure 17.- Motion of rudder and airplane under the influence of solid friction in the rudder control system.